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Pointless Spaces in General Relativity

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A new approach to quantize gravity based on the notion of differential algebra is suggested. It is shown that the differential geometry of this object cannot be described in terms of points. A spatialization procedure giving rise to points by losing part of the entire structure is discussed. The counterparts of the traditional objects of differential geometry are studied.

FOREWORD

In general relativity an event in spacetime is idealized to a point of a four-dimensional manifold. Such idealization is adequate within classical physics, but is unsatisfactory from the operationalistic point of view. In quantum theory the influence of a measuring apparatus on the object being observed cannot in principle be removed. We could expect the metric of a quantized theory to be subject to fluctuations, whereas the primary tool to separate individual events is just the metric. Thus a sort of *smearing procedure* for events is to be imposed on the theory.

An essential step in this direction was the idea to build the differential geometry in terms of abstract algebras. Geroch (1972) proposed to generalize the notion of algebra of smooth functions on a manifold to that of *Einstein algebra* whose elements are not yet functions. This generalization was successful, since the entire content of general relativity can be reformulated in such a way that the underlying spacetime manifold is used only once: to define the collection of smooth functions.

However, since the *commutative* case is considered, the absence of points is, roughly speaking, an illusion. As a matter of fact, a *commutative* algebra can always be represented by functions on an underlying space. Such a

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representation is, for instance, the Gel'fand construction (for normed algebra), which is the special case of the representation of commutative algebra on its spectrum. So, in the case of the commutative algebra, points implicitly exist. We consider this in more detail in Section 2.

The goal of this paper is to essentially remove points from the theory. Metaphorically speaking, instead of smearing *out* of events, we smear them *off*. This happens automatically when we pass to noncommutative Einstein algebras, whereas the reproduction of geometrical constructions causes a number of purely mathematical obstacles. The analysis of these problems is concluded by an example of a finite-dimensional noncommutative Einstein algebra (Section 8).

1. POINT-FREE APPROACH TO DIFFERENTIAL GEOMETRY

The emphasis of this section is the observation that the standard coordinate-free approach to the differential geometry of smooth manifolds can be thought of as (or converted to) point-free.

The basis of the differential geometry is the notion of *vector field*. It is known that any vector field v can be associated with the differential operator in the algebra $\mathcal{A}$ of smooth functions on the manifold acting as the derivation along this vector field. This operator v is linear, and its main feature is the *Leibniz rule*:

$$v(ab) = v(a)b + av(b) \tag{1.1}$$

It is known that linear operators in $\mathcal{A}$ satisfying (1.1) are exhausted by that induced by actions of vector fields. That is why the difference is not drawn between such operators and vector fields: this is the essence of the coordinate-free account of differential geometry. As a matter of fact, coordinates appear only once: to specify the algebra $\mathcal{A}$ of smooth functions, since the notion of smoothness is referred to local maps. The forthcoming notions such as connection, torsion, curvature, and others need no local coordinates in their definition.

We emphasize that at the mere level of definitions the principal notions of differential geometry require no coordinates, nor even *points*: the fact that $\mathcal{A}$ is the algebra of functions on a set is never used. Thus the global geometry per se does not confine us to a set-theoretic concept of space.

2. TOWARD NONCOMMUTATIVITY

In this section we analyze the obstacles arising in the noncommutative generalization of the algebraic construction of differential geometry.

Basic Algebra. The first question is, why are we going to fetch noncommutativity to geometry? The rough answer is that we follow the tradition of quantization. An amount of noncommutativity in the geometry itself is needed to quantize it. This produces the following problem: the lack of points in this quantum geometry requires a "spatialization" procedure to be imposed onto the general scheme to describe the *observable* entities.

We shall start with an associative and, in general, noncommutative algebra $\mathcal{A}$ over real numbers, which will play a role analogous to that of the algebra of smooth functions. It will be called the *basic algebra* of the model.

Spatialization Procedure. Let us try to extract the geometry from the basic algebra $\mathcal{A}$ on its coarsest level, that is, the set-theoretic one. As is usually done, we must consider the elements of $\mathcal{A}$ as functions defined on a certain set M, and perhaps taking values in a noncommutative domain R. That is, the representation of $\mathcal{A}$ by means of homomorphism $\hat{}$ is introduced:

$$a \mapsto \hat{a}$$

where $\hat{a}$ is a function $M \to R$. Thus each point $m \in M$ is associated with the two-sided ideal $I(m) \subseteq \mathcal{A}$:

$$I(m) = \{a \in \mathcal{A} | \hat{a}(m) = 0\}$$

Now we see that the resources of spatialization are bounded by the number of two-sided ideals in $\mathcal{A}$. If $\mathcal{A}$ contains two-sided ideals, it can be, as a rule, decomposed into mutually commuting components. So, each point can be associated with at least a simple component of the decomposition of $\mathcal{A}$. The conclusion is that *spatialization and noncommutativity are in some sense complementary*: commutation relations cannot be described in terms of points.

When the basic algebra $\mathcal{A}$ is commutative and satisfies some additional requirements (is a Banach algebra), the proposed construction is just the Gel'fand representation endowing the set M by a natural topology. So, the commutative case makes it possible to store the *topological* space M so that $\mathcal{A}$ is represented by continuous functions on M. However, the Gel'fand construction does not yield the differential structure for M.

Differential Structure. The lack of points is not an obstacle to introducing differential structure with all its attributes. As in the commutative case, it is introduced in terms of the collection Der$\mathcal{A}$ of derivations of the basic algebra $\mathcal{A}$. Recall that a *derivation* of $\mathcal{A}$ is the linear mapping v: $\mathcal{A} \to \mathcal{A}$ obeying the Leibniz rule (1.1). Der$\mathcal{A}$ is the Lie algebra over $\mathcal{R}$ with respect to the commutation

$$[u, v]a = u(va) - v(ua)$$

Scalars. In the commutative case we can multiply a vector by any element of the basic algebra $\mathcal{A}$. In general, an element $v \in \mathrm{Der}\mathcal{A}$ multiplied by an element $a \in \mathcal{A}$ does not obey the Leibniz rule. However, to define such objects such as, say, a connection, multiplicators are necessary: they play the role of *scalars*. So, we have to clarify which elements of $\mathcal{A}$ can serve as multiplicators for vectors. Evidently, each element of the center $Z(\mathcal{A})$ of the algebra $\mathcal{A}$ is suitable for this purpose: for each $z \in Z(\mathcal{A})$, $v \in \mathrm{Der}\mathcal{A}$, a, $b \in \mathcal{A}$, the Leibniz formula holds:

$$(ab) = z(v(a)b + av(b)) = (zv)(a)b + a(zv)(b)$$

In the sequel we shall confine ourselves to this class of multiplicators, that is, $\mathrm{Der}\mathcal{A}$ will be considered as a $Z(\mathcal{A})$-module. So, $Z(\mathcal{A})$ will be set up as the set of scalars:

$$S = Z(\mathcal{A})$$

Note that the set S of multiplicators for V may be essentially broader than $Z(A)$, and even noncommutative; we shall not tackle this problem in this paper since such a level of generality is not needed for the account of the proposed model.

3. DIFFERENTIAL ALGEBRAS

We introduce the notion of a *differential algebra* as a pair $(\mathcal{A}, V)$, $V \subseteq \mathrm{Der}\mathcal{A}$. The reasonable restrictions on the choice of V are analyzed in this section.

There are natural classical examples with $V \neq \mathrm{Der}\mathcal{A}$. The elements of $\mathrm{Der}\mathcal{A}$ are the direct generalization of vector fields on smooth manifolds. Sometimes, even in the classical (commutative) situation, not all vector fields are considered. For example, if the algebra of smooth vector fields on a Lie group is studied, it is natural to confine oneself to left-invariant ones. Another example is provided by dynamical systems associated with the subalgebras of $\mathrm{Der}\mathcal{A}$ with one generator. In classical mechanics, to fix up a subalgebra $V \subseteq \mathrm{Der}\mathcal{A}$ means to define the virtual shifts of the system.

Constants. Now let a subset $V \subseteq \mathrm{Der}\mathcal{A}$ be set up whose elements are thought of as "virtual infinitesimal shifts." The question immediately arises of which elements of $\mathcal{A}$ are invariant with respect to all these shifts. Call such elements *constants*. The set $\mathcal{K}$ of constants

$$\mathcal{K} = V^c = \{k \in \mathcal{A} \,|\, \forall v \in V \;\; vk = 0\} \tag{3.1}$$

is always the subalgebra of $\mathcal{A}$ (proof is straightforward). Clearly, $v(ka) =$

$k \cdot va$ for each $v \in V$, $a \in \mathcal{A}$, $k \in \mathcal{K}$. The counterpart of $\mathcal{K}$ in classical mechanics is the algebra of integrals of a dynamical system.

We emphasize that the set of constants depends substantially on the choice of V. It follows from (3.1) that $\mathcal{K} = V^c$ shrinks when V broadens. In particular, when $V = \mathrm{Der}\mathcal{A}$ we call the elements of $C_{\mathcal{A}} = (\mathrm{Der}\mathcal{A})^c$ *basic constants*. $C_{\mathcal{A}}$ lies in all other algebras of constants: $C_{\mathcal{A}} \subseteq V^c$. When $\mathcal{A}$ is not commutative, $C_{\mathcal{A}}$ is nevertheless commutative and, moreover, is contained in the center $Z(\mathcal{A})$ of $\mathcal{A}$. Note that $C_{\mathcal{A}}$ always contains the elements of the form $\lambda \circ 1$, $\lambda \in \mathcal{R}$. For any $u, v \in \mathrm{Der}\mathcal{A}$, $c \in C_{\mathcal{A}}$, $a \in \mathcal{A}$,

$$[cu, v]a = cuv(a) - v(cu(a)) = cuv(a) - vc \cdot u(a) - cvu(a) = c[u, v]a$$

Therefore

$$[cu, v] = c[u, v] = [u, cv] \tag{3.2}$$

(the second equality is proved likewise). Hence, $\mathrm{Der}\mathcal{A}$ may be thought of as the Lie algebra over $C_{\mathcal{A}}$. The following example shows that $C_{\mathcal{A}}$ may be broader than $\mathcal{R}$.

Example 3.1. Let $\mathcal{A}$ be the (commutative) algebra of C^∞-functions on a smooth manifold M. Then $\mathrm{Der}\mathcal{A}$ is the Lie algebra of smooth vector fields on M. In this case $C_{\mathcal{A}}$ is the algebra of continuous *locally* constant functions on M. The dimension of $C_{\mathcal{A}}$ is then the number of connected components of M. So, $C_{\mathcal{A}} = \mathcal{R}$ only if M is connected.

Vectors. Consider all such $u \in \mathrm{Der}\mathcal{A}$ for which V^c serves as the set of constants; denote it V^{cc}. Clearly $V \subseteq V^{cc}$; however, this inclusion may be strict. In the sequel we shall consider such collections V of vectors that are uniquely determined by their set of constants:

$$V = V^{cc} \tag{3.3}$$

Such a requirement looks reasonable since in this case V is automatically the Lie subalgebra of $\mathrm{Der}\mathcal{A}$. We shall essentially use this condition in the sequel (Section 6).

It follows from (3.3) that we could define the differential algebra as a pair $(\mathcal{A}, \mathcal{K})$ putting $V = \mathcal{K}^c$, precisely as was proposed by Geroch (1972).

4. CONNECTION AND CURVATURE

In this section we generalize connection to noncommutative differential algebras and introduce torsion and curvature.

Connection. In classical differential geometry connection provides the means to form the derivative of a vector (field) along another one. We shall define it as a V-valued function $\nabla_x y$ of two arguments $x, y \in V$ such that it is:

1. S-linear by the lower argument: $\nabla_{zx}y = z\nabla_{xy}$.
2. $C_{\mathcal{A}}$-linear by the upper argument: $\nabla_x(cy) = c\nabla_x y$.
3. Derivative with respect to x:

$$\nabla_x(zy) = x(z)\cdot y + z\nabla_x y \tag{4.1}$$

(recall that S, $C_{\mathcal{A}}$ are the sets of scalars and basic constants, respectively).

Torsion and Curvature. As in the classical situation, the *torsion* is defined as the V-valued function

$$T(x, y) = \nabla_x y - \nabla_y x - [x, y], \qquad x, y \in V \tag{4.2}$$

It can be checked directly that T is S-bilinear. The *curvature* is defined as follows:

$$R(u, x)y = \nabla_u\nabla_x y - \nabla_x\nabla_u y - \nabla_{[u,x]}y \tag{4.3}$$

R is the V-valued function of three arguments x, y, $u \in V$. It can be also verified that it is S-trilinear.

Ricci Curvature. In classical geometry the Ricci curvature is formed as a contraction of the curvature (4.3). Let us consider it in more detail. Fixing up the values x, y in (4.3), we obtain the family of S-linear operators $\mathcal{R}_{xy}$: $V \to V$,

$$\mathcal{R}_{xy}u = R(u, x)y \tag{4.4}$$

In the case when the notion of trace is meaningful for operators in V, the *Ricci curvature* is defined as the trace of each operator $\mathcal{R}_{xy}$,

$$\mathrm{Ric}(x, y) = \mathrm{Tr}\,\mathcal{R}_{xy} \tag{4.5}$$

The trace problem is to define the trace in the general situation as an S-linear scalar-valued functional on some class of linear operators in V such that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

When V possesses a basis, the trace is defined as the trace of the appropriate matrix. However, even the module of vector fields V may not have a basis at all. For instance, in the case of a 2-dimensional sphere $\mathcal{S}^2$, any smooth vector field on $\mathcal{S}^2$ has at least one point where it vanishes. Hence no pair of vector fields can form the basis (but any three vector fields on $\mathcal{S}^2$ are linearly dependent). In classical geometry we are always in a position to localize the situation so that at *each point* the operator $\mathcal{R}_{xy}$ is represented by the matrix in a basis of the tangent space, so that the trace is well defined.

In the pointless situation all constructions are global. So, to solve the trace problem along these lines, the basis needs to be set up. There are two obstacles, functional and algebraic, to doing it. The former is that the module V may contain infinitely many independent elements; the latter is that even a finitely generated module may not have a basis. A possible way to avoid these problems is the implementation of algebraic localization (Atiyah and Macdonald, 1969).

Another approach to the trace problem is inspired by conventional Riemannian geometry. It is based on the canonical isomorphism

$$V \otimes V^* \cong \mathcal{L}(V) \tag{4.6}$$

where V^* is the space of covector fields, and $\mathcal{L}(V)$ is the space of linear (w.r.t. scalars) operators in V. A linear operator (4.4) represented as the element of $V \otimes V^*$ is decomposed into the sum of terms of the form $v \otimes w$. The trace of each term is $w(v)$, hence the trace of the entire operator is the sum of appropriate values: for $T \in \mathcal{L}(V)$

$$T = \sum v_i \otimes w_i \qquad \text{and} \qquad \mathrm{Tr}(T) = \sum w_i(v_i) \tag{4.7}$$

In the general situation a more thorough treatment of the dual space is needed.

5. COVECTORS

Consider in more detail the algebraic structure of the set of vectors V. First of all, V is the real vector space. Besides, it is equipped with the structure of a two-sided S-module, where S is the set of scalars, which is assumed to be the center $Z(\mathcal{A})$ of the basic algebra $\mathcal{A}$: for each $s \in S$, $a \in \mathcal{A}$, $v \in V$,

$$a = s(va) = (v \cdot s)a = (va) \cdot s$$

Note that the condition (3.3) is essential: otherwise V would not possess the S-module structure.

Covectors. Now introduce the set of *covectors* V^+ as

$$V^+ = \mathrm{Hom}(V, \mathcal{A})$$

the set of all S-homomorphisms from the S-module V to $\mathcal{A}$ considered S-module. V^+ is also the real vector space and possesses the natural structure of an $\mathcal{A}$-bimodule: for each $w \in V^+$, $v \in V$, $a \in \mathcal{A}$,

$$(w \cdot a)(v) = w(v)a$$

$$(a \cdot w)(v) = aw(v)$$

The discrepancy between a bimodule and a two-sided module is that for arbitrary $a \in \mathcal{A}$, $w \in V^+$, $wa \neq aw$. However, in bimodules the following holds:

$$(aw)b = a(wb), \qquad a, b \in \mathcal{A}, \quad w \in V^+$$

Cartan Differentials. To each element $a \in \mathcal{A}$ a covector $da \in V^+$ is canonically associated:

$$da(v) = v(a) \tag{5.1}$$

The operator d acts from $\mathcal{A}$ to V^+ (both considered $\mathcal{A}$-bimodules) so that the Leibniz rule holds:

$$d(ab) = da \cdot b + a \cdot db \tag{5.2}$$

It is the set of constants V^c which is the kernel of the operator d. On the other hand, not any covector may be of the form da for some $a \in \mathcal{A}$. For each $w \in V^+$ define its *Cartan differential* dw as the following skew-symmetric bilinear form on V:

$$(dw)(v_1, v_2) = v_1 w(v_2) - v_2 w(v_1) - w([v_1, v_2]), \qquad v_1, v_2 \in V$$

When $w = da$ for some $a \in \mathcal{A}$, dw is necessarily equal to zero. However, this is not a sufficient condition.

De Rham Cohomologies. A differential form w is called *exact* if $w = da$ for some $a \in \mathcal{A}$, and *closed* if $dw = 0$. Since $dda = 0$, each exact form is closed. Both exact and closed forms are submodules of V^+, hence their quotient can be formed, called the module of one-dimensional *De Rham cohomologies* $\mathcal{H}^1_{\mathcal{A}}(V)$. In the classical case it depends on the topology of the underlying manifold (e.g., it is zero for simply connected manifolds). In our theory, it remains the structural characteristic of the differential algebra $(\mathcal{A}, V)$.

The 0-dimensional cohomologies are defined as the algebra V^c of constants. The closedness condition is now referred to the elements of $\mathcal{A}$: $da = 0$. In virtue of (5.1) that means that $va = 0$ for each $v \in V$. In the classical case $\mathcal{H}^0_{\mathcal{A}}(V)$ is the number of connected components of the manifold M (see Example 3.1).

Coupling. There is canonical coupling between V and V^+:

$$\langle v, w \rangle = w(v), \qquad v \in V, \quad w \in V^+ \tag{5.3}$$

Due to noncommutativity we have to take care of the order of factors:

$$\langle v, aw \rangle = a\langle v, w \rangle; \qquad \langle v, wa \rangle = \langle v, w \rangle a, \qquad a \in \mathcal{A}, \quad v \in V, \quad w \in V^+$$

The form $\langle *, * \rangle$ is S-linear by the first and $\mathcal{A}$-linear by the second argument. Thus any $v \in V$ can be considered as an $\mathcal{A}$-linear form on V^+:

$$v \mapsto \langle v, * \rangle \tag{5.4}$$

In the classical theory all $\mathcal{A}$-linear functionals on V^+ are exhausted by that of the form (5.4), which does not hold in the general case. We shall deal with the class of *regular differential algebras* for which this also holds.

Scalar Covectors. As already mentioned, the set V is the S-module. Its dual module is the set V^* of all *S-valued forms* on V. It is natural to call the elements of V^* *scalar covectors*. Clearly $V^* \subseteq V^+$, and moreover, V^* is the S-submodule of V^+.

The canonical coupling (5.3) between V and V^* makes it possible to consider the elements of V as S-linear forms (5.4) on V^*. Note that each $\mathscr{A}$-linear form is S-linear, but not vice versa, hence the regularity requirement does not ensure that any S-linear form on V^* is induced by some $v \in V$. The module V is called *reflexive* whenever $V = V^{**}$.

The Trace Problem Again. Let us return to the problem of representability of linear operators by sums of terms $w(v)$. We cannot expect that the formula (4.6) will hold for V^+ since V^+ is broader than the "real" dual V^*. Since V^* is the submodule of V^+, (4.6) becomes *embedding*:

$$\mathscr{L}(V) \cong V \otimes V^* \subseteq V \otimes V^+ \tag{5.5}$$

whenever V is reflexive: $V = V^{**}$. In this case the trace is also well-defined in accordance with (4.7).

However, the regularity of $(\mathscr{A}, V)$ does not imply the reflexivity of V. When V is not reflexive, the elements of $\mathscr{L}(V)$ are *approximated* by the elements of the tensor product $V \otimes V^*$. To solve this problem, some additional structure on V or $\mathscr{A}$ must be imposed such as norm or topology. However, the trace remains nonuniquely defined: at least up to a constant factor. This nonuniqueness may change the form of the Einstein equation even in the classical situation (cf. Section 7).

6. METRIC STRUCTURE

The metric structure is introduced by defining an S-bilinear $\mathscr{A}$-valued symmetric form $g(u, v)$ on V. It immediately induces the operator $V \to V^+$ defined for any $v \in V$ as $v \mapsto \underline{v}$ such that

$$\underline{v}(u) = g(u, v)$$

We shall require the nondegeneracy of g; hence the mapping $v \mapsto \underline{v}$ will be the injection. In the classical situation it is an isomorphism. For general differential algebras $v \mapsto \underline{v}$ is a mere embedding.

Gradients. For further purposes (to introduce the Levi-Civita connection) we shall use a weaker constraint than the requirement of isomorphism $V \cong V^+$. Namely

$$\forall a \in \mathscr{A} \quad \exists v \in V \quad \underline{v} = da \tag{6.1}$$

We shall call this vector $\underline{v}$ the *gradient* of the element $a \in \mathcal{A}$ and denote it

$$\underline{v} = \text{grad } a \qquad \text{iff} \quad \underline{v} = da$$

The notion of a gradient is unambiguously defined in virtue of the nondegeneracy of the metric form g.

Levi-Civita Connection. In the standard version of general relativity the Levi-Civita connection is used. That is, when the metric g is set up, the two following conditions for the connection ∇ hold for all $u, x, y \in V$:

$$\nabla_x y - \nabla_y x = [x, y] \tag{6.2}$$

$$u(g(v, x)) = g(\nabla_u v, x) + g(v, \ \nabla_u x) \tag{6.3}$$

The condition (6.2) means that ∇ is torsion-free: $T = 0$, and (6.3) means that the covariant derivative of g is zero.

In classical geometry the Levi-Civita connection is uniquely defined by the metric and always exists. Returning to the general situation, let us try to build the connection associated with the metric g. First suppose it exists. Recall how the values of Christoffel symbols are obtained in classical geometry. The variables u, v, x are cyclically permuted in (6.3), which yields, using (6.2),

$$\begin{aligned} 2g(x, \nabla_v u) &= u(g(v, x)) + x(g(u, v)) - v(g(x, u)) \\ &\quad - (g(u, [v, x]) + g(x, [u, v]) - g(v, [x, u])) \end{aligned} \tag{6.4}$$

To prove the existence, denote by $\Gamma(u, v, x)$ the right side of (6.4). Then fix up $u, v \in V$ and consider the function $D_v u$: $\mathcal{A} \to \mathcal{A}$ defined as

$$D_v u(a) = \Gamma(u, v, \ \text{grad } a)$$

It can be checked directly that $D_v u$ obeys the Leibniz rule (4.1) and annihilates each constant from V^c; hence the mapping $a \mapsto D_v u(a)$ is really an element of V. Moreover, the mapping $(v, u) \mapsto D_v u$ satisfies the definition of connection (Section 4).

Now we see the role of the conditions (3.3) and (6.1): they enable the validity of the existence theorem for the Levi-Civita connection in differential algebras. The uniqueness follows from the nondegeneracy of g.

7. EINSTEIN EQUATION

Now we have everything to introduce the point-free counterpart of the Einstein equation. Conventional theory postulates the equality between the Einstein tensor depending on geometry only and the momentum-energy tensor.

To form the left side, first introduce the analog of the scalar curvature R. In classical geometry R is the contraction of the contravariant metric tensor with the Ricci tensor. In differential algebras we have neither contraction nor tensors, but only operators. However, we have the trace of operators at our disposal. This worked already when the Ricci operator was defined as the trace of the Riemann curvature (4.5).

The Ricci Operator and Scalar Curvature. In (4.5) the S-bilinear form $\mathrm{Ric}(x, y)$ was defined. To define the scalar curvature, we must be in a position to associate the form Ric with an operator $\mathcal{R}$ in V such that

$$\mathrm{Ric}(u, v) = g(\mathcal{R}u, v) \tag{7.1}$$

Then the trace of $\mathcal{R}$ will be the scalar curvature r

$$r = \mathrm{Tr}\,\mathcal{R} \tag{7.2}$$

However, (7.2) is well defined only if (i) this operator $\mathcal{R}$ exists and (ii) $\mathcal{R}$ will be a trace-class operator w.r.t. the trace Tr. Leaving apart item (ii), we suggest a sufficient condition for $\mathcal{R}$ to exist. For each $u \in V$ define the covector $u_{\mathcal{R}}$ as

$$u_{\mathcal{R}}(v) = \mathrm{Ric}(u, v) \tag{7.3}$$

Now if we require for any $u \in V$ the existence of $v \in V$ such that

$$u_{\mathcal{R}} = \underline{v} \tag{7.4}$$

the operator $\mathcal{R}$ can be immediately defined as $\mathcal{R}u = u_{\mathcal{R}}$.

The Einstein Equation. In conventional relativity the operator form of the Einstein equation is

$$R^i_k - \tfrac{1}{2}R\delta^i_k = \kappa T^i_k \tag{7.5}$$

In differential algebras R^i_k becomes the Ricci operator $\mathcal{R}$ and the scalar curvature is r. So, everything is now ready to write the analog of (7.5):

$$\mathcal{R} - \tfrac{1}{2}rI = \kappa\mathcal{T} \tag{7.6}$$

Note that (7.6) substantially depends on the choice of the trace; however, so does (7.5)! It is assumed in the classical case that the trace of the unit operator δ^i_k is equal to 4 (the dimension of the spacetime manifold). However, we could redefine the trace so that all contractions would be multiplied by a constant α, and the factor 1/2 in (7.5) will become $1/2\alpha$.

We reproduce the Einstein equation (7.5) in the form (7.6), which requires the introduction of the momentum operator $\mathcal{T}$, which acts as follows. Recall that V is interpreted as the set of virtual shifts, so, if $v \in V$ is associated with a shift of the observer, $\mathcal{T}v$ yields the energy flow seen by the observer.

8. AN EXAMPLE

Consider the basic algebra $\mathcal{A} = \mathrm{Mat}_4(\mathfrak{R})$ of square 4×4 matrices, and let $\mathcal{K}$ be the subalgebra of $\mathcal{A}$ generated by the matrices

$$\mathbf{e}_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{e}_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.1}$$

Denote by $\mathbf{e}_{ik}$ the matrix having 1 at the (i, k) entry with all other entries equal to zero. In accordance with (3.3), form the set $V = \mathcal{K}^c$. Any derivative in $\mathrm{Mat}_n(\mathcal{A})$ is inner, that is, each element of $v \in V$ is associated with a matrix $v \in \mathcal{A}$ so that $v(a) \equiv va - av = [v, a]$ for any $a \in \mathcal{A}$. The Lie operation in V is the commutator of appropriate associated matrices.

It can be checked by direct calculation that V is the 6-dimensional Lie algebra spanned on the elements $\{\mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{33}, \mathbf{e}_{34}\}$. The nonzero commutators of the basis elements are

$$[\mathbf{e}_{13}, \mathbf{e}_{33}] = \mathbf{e}_{13}; \qquad [\mathbf{e}_{13}, \mathbf{e}_{34}] = \mathbf{e}_{14}$$

$$[\mathbf{e}_{23}, \mathbf{e}_{33}] = \mathbf{e}_{23}; \qquad [\mathbf{e}_{23}, \mathbf{e}_{34}] = \mathbf{e}_{24}; \qquad [\mathbf{e}_{33}, \mathbf{e}_{34}] = \mathbf{e}_{34}$$

and other commutators are zero. The set of scalars S of the model will be the set of scalar matrices λI, where I is the unit matrix and $\lambda \in \mathfrak{R}$.

Suppose that some metric structure on $(\mathcal{A}, V)$ is defined: g: $V \times V \to \mathcal{A}$. In accordance with (7.1) the values of g must be the values of the trace, i.e., scalars. Hence $g(u, v) = \lambda I$.

Now we try to build the Levi-Civita connection associated with the metric g. To do this, we must check the existence of gradients (6.1). That means that for all $a \in \mathcal{A}$ such $v \in V$ must exist that for all $u \in V$

$$g(v, u) = da(u) = u(a) = [u, a]$$

Hence the commutator $[u, a]$ is a multiple of the unit matrix. This is possible in *no finite-dimensional* case. However, it may be possible in infinite-dimensional space where canonically conjugate variables do exist. Nevertheless, torsion-free connections exist in our example, for instance, that defined as

$$\nabla_u v = uv$$

It can be checked directly that the conditions (4.1) are valid due to the fact that the scalars are multiples of the unit matrix.

This example shows that the *affine* differential structures can survive even on finite-dimensional basic algebras, while the attempts to build the noncommutative *Riemannian* geometry require the infinite dimensionality of basic algebras. Moreover, any attempts to substitute spacetime by finitary patterns (Dubois-Violette, 1988; Zapatrin, 1993a) can restore either metric (Regge calculus) or topology, but not both at once.

9. SUMMARY AND CONCLUDING REMARKS

We begin with an outline of our main results.

Models of point-free differential geometries have been proposed (Section 3) as pairs $(\mathcal{A}, V)$, called *differential algebras*, which are the noncommutative generalization of Einstein algebras (Geroch, 1972). The substantial feature of noncommutativity is the discrepancy between the elements of the basic algebra (analog of the smooth functions) and scalars (Section 2).

Luckily, the geometry of affine connection survives in noncommutative differential algebras, including the notions of torsion and curvature (Section 4). It is even possible to introduce "topological" invariants such as De Rham cohomologies (Section 5). The conditions for the Ricci form to exist were reduced to the trace problem. Possible ways to solve it were shown.

The conditions for a metric structure to be definable were studied, giving rise to the notion of *regular* differential algebras (Section 6). It happens that the Levi-Civita connection (which can always be restored from the symmetrical metric form in the classical case) may not exist in the noncommutative case (example in Section 8): it depends on the possibility to build gradients (Section 6). However, if it exists, it is still unique.

The scalar curvature can also be defined under certain circumstances. If it becomes possible, the operator analog of Einstein equation is introduced. It is shown that it does not depend on the normalization of the trace.

The idea to consider vectors as differential operators applied to functional algebras, but defined on a broader class of spaces than manifolds, called *differential spaces*, was used to implement spaces with singularities to general relativity. Heller *et al.* (1989) showed that a reasonable definition of differential structure can be formulated in terms of a certain algebra $\mathcal{A}$ of functions so that even the analog of Lorentz structure can be introduced (Multarzinski and Heller, 1990). In particular, when $\mathcal{A}$ is an algebra of smooth functions on a manifold, the standard differential geometry is restored.

The approach we suggest can be considered as a reasonable way to quantize gravity. The main problem arising here is to find the appropriate representations of the basic algebras. With regard to the source of basic algebras, one has Wheeler's suggestion to consider logic as pregeometry, which could work here. The first step along these lines was made by Isham

(1989): the lattice of all topologies over a set was considered, and the analog of creation and annihilation operators was suggested. The appropriate algebra could be taken as a basic one. Moreover, starting from an arbitrary property lattice as background object, one can always build the semigroup [called *generating* (Zapatrin, 1993b)] whose annihilator lattice restores this property lattice. Then the algebra spanned on this semigroup could play the role of the basic algebra $\mathcal{A}$.

The next step is the *spatialization* procedure (mentioned in Section 2). When a differential structure V and a metric g are set up, the problem arises to extract usual (i.e., point) geometry from the triple ($\mathcal{A}$, V, g). To return to points, we must consider a subalgebra $\mathcal{C} \subseteq \mathcal{A}$ such that $\mathcal{C}$ would be commutative (to enable functional representation) and in some sense concerted with V and g. We have not yet tackled this problem of *eigensubalgebras* in detail, although it looks like a direct way to reveal events within our scheme. It is noteworthy that whenever the triple ($\mathcal{A}$, V, g) is set up, there still may exist several functionally representable eigensubalgebras associated with possibly nonisomorphic geometries. This means that *the observed geometry depends on observation*, which is in complete accordance with the quantum mechanical point of view.

Finally, we should mention that among various approaches to noncommutative geometry the closest one to ours is that proposed by Dubois-Violette (1988): in our terms, he works only with $V = \mathrm{Der}\mathcal{A}$.

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